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”Evolutionary Game Theory: Imitation in the *Hotelling Game*”

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Introduction

Vega-Redondo (1997*b*) has shown in his well-known paper "The Evolution of Walrasian behavior" that imitation in a simple *Cournot Game* leads to competitive prices which are also known as *Walrasian prices*. The intuition behind this is quite simple: As every firm is confronted with the same price[1], the firm which offers the highest quantity will also realize the highest profit. Therefore, every other firm will imitate the quantity of that respective firm which consequently leads to a higher quantity overall, hence the prices will decrease. This imitation dynamic (with mutations) proceeds until the *Walrasian price* is reached.

We will also focus on imitation behavior of economic agents throughout this study. One of the most important advantages of *imitation of the most successful* agent is, that it is a very simple decision rule which only requires the knowledge of the action and the success of the other agents as information pool. In complex environments imitation could be the only plausible and available decision rule. Not only is imitation an essential mechanism[2] in nature but also a lot of human decisions can be traced back to it. Economic markets can likewise constitute similarly complex environments in which imitation can be used as decision rule or at least as a rule of thumb. In Duersch, Oechssler & Schipper (2010) it is shown that imitation of the best is an 'unbeatable'[3] decision heuristic. In that paper the authors mean by unbeatable that the imitation behavior of the agents can not be exploited by an agent who is aware of the fact that the others are imitating.

Vega-Redondo used the graph theoretic technique introduced by Freidlin & Wentzell (1984) which was established as a solution concept for games in

[1]We assume here that the price is higher than the costs of producing one good, but in the other case the dynamics would also lead to the same price.

[2]An example would be replication in the evolution process.

[3]For this to become valid some assumptions have to be fulfilled, symmetry of the players for example.

economics by Kandori, Mailath & Rob (1993) [4]. This approach uses a *Markov Chain* to represent the dynamic process. Further, it introduces mutations[5] into the dynamics in order to be able to determine *Long Run Equilibria*. We will use this dynamical approach as well to determine the *Long Run Equilibria* in the *Markov Chain* with 'imitating the best' dynamics. We will call these *Long Run Equilibria* in such dynamics *Imitation Equilibria*, thus we use a different definition for the term *Imitation Equilibrium* as it has been used by Selten & Ostmann (2001).

Instead of the *Cournot Game* it will be the *Hotelling Game*[6] we are going to use in our study. In this game two firms are exposed to a price competition. The firms will use imitation as a price setting rule, so they will simply set the price of the firm, that had the highest profit in the previous round. As mentioned in Apesteguia, Huck & Oechssler (2007)[7] imitation of one's own opponents leads to more competitive outcomes than the *Nash Equilibrium*, thus we are also getting more competitive prices in our setting. However we are not observing marginal-cost prices which are obtained as *Imitation Equilibrium* in the *Cournot Game* by Vega-Redondo. We show that the *Imitation Equilibrium* in the *Hotelling Game* will be exactly the mean of the marginal-cost price and the *Nash Equilibrium* price.

Selten & Apesteguia (2005) have taken a similar approach for price competition on the circle. In their analysis they are focusing on experimental results. By these means they are trying to get some empirical evidence of imitation behavior of the involved individuals. Ultimately, they stress that "imitation is clearly present in the behavior of those subjects"[8], which empirically strengthens the need to investigate more on the economic implications of imitation.

In Alós-Ferrer & Ania (2005) symmetric aggregate games have been investigated. The *Hotelling Game* represents such a game. Therefore, we can and will use some of the results of this paper, in particular the link between *Evolutionary Stable Strategies* and the *Imitation Equilibrium*.

In the first chapter we will introduce the *Hotelling Game* with the usually associated assumptions and calculate the basic characteristics (like the *Nash Equilibrium*) of that game. These results will be used later on as a benchmark

[4] Young (1993) is another famous economic application of this approach. Also, Ellison (1993) should be mentioned for applying it on *Local Interactions*.

[5] Here mutation is referred to random selection of actions which occurs with a small probability.

[6] Definition can be found in Tirole (1988, 2.1.2), original in Hotelling (1929)

[7] As Vega-Redondo they analyzed games in the *Cournot* setting.

[8] Selten & Apesteguia (2005, p. 185)

for the imitation results.

Subsequently, in the second chapter, we will analyze the impact of the agents' imitation behavior on the equilibrium. Hence, we will justify imitation as a decision rule as well as briefly present the *Freidlin and Wentzell* approach to model imitation behavior and calculate the unique *Imitation Equilibrium* in the introduced simple model. At the end of this chapter we will briefly describe *Relative Payoff Maximization* as another possible approach to determine the *Imitation Equilibrium* directly.

In the third chapter we will loosen up some of the assumptions made in chapter 1 and calculate the *Imitation Equilibria* in these modified models. On the one hand we will alter the rigorous cost structure of the simple model. Among other modifications we will introduce asymmetric marginal costs as well as fixed costs. On the other hand we will modify the distribution of the costumers. Thereby we will see that some basic assumptions are required to obtain a unique *Imitation Equilibrium*.

In the ultimate part of the thesis we will finally illustrate the obtained results by simulating the game and illustrating the probability distributions. In this process we will also focus on the convergence of the *Imitation Equilibria* in respect to the mutation rate and the price grid size.

Chapter 1

Simple Model

We start by introducing a first simple model. We will give different economic interpretations of the game and calculate the *Nash Equilibrium* and the resulting profits. These results will serve us later on as benchmarks for the *Imitation Equilibrium*.

1.1 The Model

As our basic game we will use a simplified version of the *Hotelling Game*. This game was introduced by Hotelling (1929)[1].

We chose the *Hotelling Game* as underlying game, because this game equips the economic agents in the *Bertrand Game* with some market power. Such circumstances let this model represent real world markets better than the pure *Bertand Game* while still being mathematically easily manageable.

In the *Hotelling Game* there are two firms $I := \{1, 2\}$ which represent the players (firms) of the game. The two firms are living in a one-dimensional space, the unit interval. Each firm is located at one end of this unit interval. Contrary the usually made assumption on movements we do not allow movements in our model.

[1]Hotelling has interestingly shown among other results that "an undue tendency for competitors to imitate each other in quality of goods, in location, and in other essential ways"(Hotelling (1929, p. 41)) exists. We want to stress that the sort of imitation mentioned by the author is different to our imitation approach. The imitation mentioned by Hotelling is a result of rational behavior in the *Hotelling Game* whereas we use imitation as an 'irrational' behavioral approach (only irrational if there are no information constraints) for solving games.

1.1.1 Consumer distribution

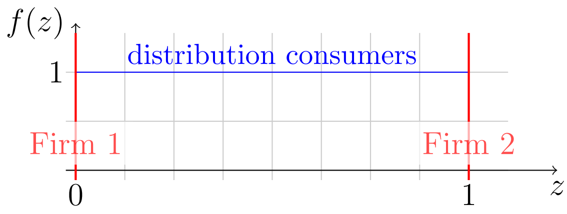


Figure 1.1: Distribution in the simple model

In the model there are infinitely many consumers who are uniformly distributed on the unit interval (figure 1.1). This uniform distribution can be formalized as a probability density function $f(x)$ (x denotes the location at the unit interval) defined as:

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

$F(x)$ will denote the cumulative distribution function of the density function $f(x)$, which is defined by $F(x) := \int_{-\infty}^{x} f(t)dt$.

1.1.2 Costs

Production Costs

Both firms produce identical goods. In the simple model we assume that both firms do not have fixed costs $f_1 = f_2 = 0$. Furthermore, the two firms have identical positive constant marginal costs $c_1 = c_2 = c \in \mathbb{R}_+$. These assumptions about the cost structure will also be loosened up in chapter 3.

Transport Costs

The consumers have unit demand which means they buy exactly one of the two produced goods. For buying the good each consumer has to travel on the unit interval to the firm and buy it there. Traveling causes the consumer transport costs $t(d)$ which depend on the distance d between the location of the consumer to the firm. Let us assume that the transport costs are linear in respect to the distance, so $tc(d) = td$, where $t \in \mathbb{R}_{++}$ denotes the costs of traveling one unit (from one end to the other of the unit interval).

Product Differentiation

There is also another interpretation[2] of the 'transport' costs which the consumers have to bear and are dependent on the location of the consumer.

We could assume that the firms do not offer exactly the same good. Since the consumers have different tastes the firms horizontally differentiate[3] their products to address certain fractions of the consumers. In this interpretation the unit interval in the model represents the possible product space. Hence the products of the firms and not the firms themselves are now located on the ends of the unit interval. The position of the consumer represents the ideal product for the consumer. The distance between the position of the consumer and the offered products by the firms can now be interpreted as difference between the desired, optimal good of the consumer and the properties of the actual offered goods by the firms.

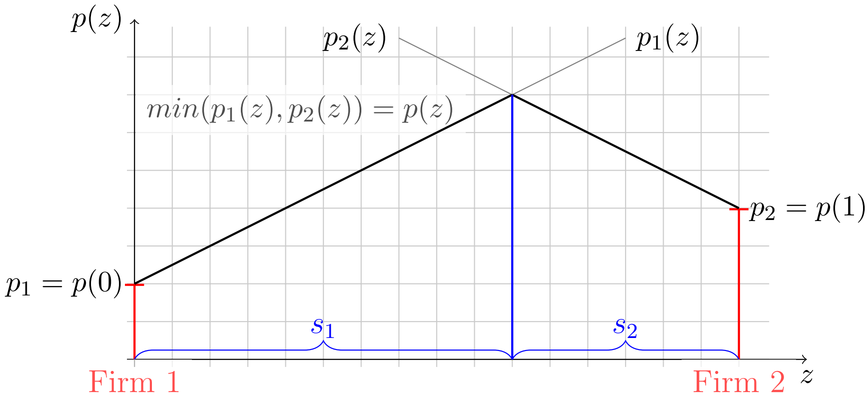


Figure 1.2: The market shares of two firms with given prices

1.1.3 Market Share and Profit

Each firm i sets a price p_i for the product. The consumers will buy the product from the firm which offers the cheaper price including the transport costs to get there and buy the good.

[2]See Mas-Colell, Whinston & Green (1995, 12) or Tirole (1988, 2.1.2), also in the original paper Hotelling (1929)

[3]Vertical differentiation would imply that the consumers agree on a preference order over the products.

Market Share

Due to the linear transport costs the consumers on the unit interval get partitioned in two connected areas. Every consumer in an area buys the good at the same firm. These areas can therefore be seen as the market shares of the firms. We will denote the market share by s_i for firm i. Figure 1.2 provides a simple example of possible market shares. In the case in which the prices of both firms differ more than the transport cost for one unit (mathematically $|p_i - p_j| > t$) the length of the area and hence the market share of the firm with the higher price would be zero whereas the area of the other firm would be the whole unit interval.

Formally we can introduce a market share function which depends on both prices and is defined as follows:

$$s_i(p_i, p_j) = \begin{cases} 0 & \text{if } p_i - p_j > t \\ 1 & \text{if } p_j - p_i > t \\ \frac{p_j + t - p_i}{2t} & \text{else} \end{cases} \tag{1.1}$$

This market share function can be derived by calculating the intersection of the two firms' price lines (depicted also in figure 1.2). The equation of the price lines are simply: $p = p_i + xt$ and $p = p_j + (1 - x)t$. As we are assuming that $t > 0$ and $t \in \mathbb{R}$ there must exist an intersection point of these two price lines. This intersection point can lie outside of the unit interval. In this case a function that is not piecewise defined would return a value higher than 1 or less than 0. Therefore we have to use a piecewise definition for the market share function.

Profit

The profit can be calculated as the marginal profit for each consumer times the amount of consumers that are buying the product. Since we have infinitely many consumers we can not use the amount of consumers. Therefore we have to use the integral over all buying consumers:

$$\pi_i(p_i, p_j) = \int_0^1 (p_i - c)\mathbf{1}_{\{p_i + xt < p_j + (1-x)t\}} dF(x) \tag{1.2}$$

The integral only integrates over a connected area, thus this expression can be simplified by using the introduced market share function $s_i(p_i, p_j)$:

$$\pi_i(p_i, p_j) = \int\limits_0^{s_i(p_i,p_j)} (p_i - c) dF(x) \tag{1.3}$$

A further step of simplification can be achieved because of the trivial form of the used consumer distribution in the model. On the unit interval the cumulative distribution function is simply the identity.

$$\pi_i(p_i, p_j) = \int\limits_0^{s_i(p_i,p_j)} (p_i - c) dx = (p_i - c) s_i(p_i, p_j)$$

So the profit of firm i is simply the market share s_i times the marginal profit of the firm. Substituting $s_i(p_i, p_j)$ with (1.1) leads to:

$$\pi_i(p_i, p_j) = \frac{1}{2t}(p_i - c)(p_j + t - p_i) \tag{1.4}$$

We have to keep in mind that we do not list the piecewise definition of the market share function here, so this profit formula only holds for $|p_i - p_j| \leq t$. If $|p_i - p_j| > t$ holds the profit of the firm with the lower price would be simply $(p_i - c)$ and the profit of the firm with the higher price would be zero.

1.2 Nash Equilibrium

If both firms are fully informed about the game[4], believe in common rationality, and want to maximize the own profit the *Nash Equilibrium* would be the appropriate solution concept to predict the actions of the firms. As a benchmark we first consider the *Nash Equilibrium* of the game.

To obtain the *Nash Equilibrium* we will first calculate the *best response* functions of the firms. Each firm maximizes the profit by setting the price (profit maximizing price) depending on the other firm's price:

$$\underset{p_i}{\operatorname{argmax}}\, \pi_i(p_i, p_j) = \underset{p_i}{\operatorname{argmax}}\, \frac{1}{2t}(p_i - c)(p_j + t - p_i)$$

[4]This includes the own and the other firm's cost structure as well as the consumer decision rule.

To get the *best response* functions we calculate the derivative of the profit function of firm i with respect to the own price:

$$\frac{d\pi_i(p_i, p_j)}{dp_i} = \frac{1}{2t}(p_j - 2p_i + c + t)$$

Setting this expression to zero we obtain the *best response* function.[5]

$$0 = \frac{1}{2t}(p_j - 2p_i + c + t)$$

We get the following best response functions for both firms. We do not have to calculate the *best response* function of the second firm separately because the game is symmetric.

$$p_1(p_2) = \frac{p_2 + c + t}{2} \qquad p_2(p_1) = \frac{p_1 + c + t}{2} \tag{1.5}$$

Using both formula in 1.5 we obtain by solving them for the prices the symmetric *Nash Equilibrium* price(s):

$$p_i = t + c \quad \forall i \in I \tag{1.6}$$

Note that also this is the only *Nash Equilibrium* if the price differences are higher than the transport costs ($|p_i - p_j| > t$).[6]

We observe that the firms set the prices in the *Nash Equilibrium* above the marginal costs. Due to the transport costs the firms gain some market power and therefore can set the prices higher than in the pure *Bertrand Game*. Such prices imply positive profits:

$$\pi_i(p_i, p_j) = \frac{1}{2t}(p_i - c)(p_j + t - p_i)$$
$$\text{with } p_i = p_j = t + c$$
$$\pi_i = \frac{1}{2}t$$

These obtained results will serve as benchmarks for the *Imitation Equilibrium* in the following chapters.

[5] The second derivative leads to $-\frac{2}{t}$ which is strict negative because we have assumed strict positive transport costs. So the profit function is concave and therefore we getting the maximizer by setting the first derivative to zero.

[6] Since we have only considered price differences less or equal the transport costs ($|p_i - p_j| \leq t$) in the analysis, we could have created an artificial equilibrium or lost one by considering only such price differences.

Chapter 2

Imitation Equilibrium

The informational requirements for the players to be able to obtain the *Nash Equilibrium* are quite strict. In this solution concept the players need to know the whole structure of the game (for example the cost structure of the other firm) as well as they have to believe in common rationality. In real life situations, the structure of games in which firms interact is highly sophisticated and complex. The firms usually do not even know the exact consumers' decision rules or the influence of their price decision on the environment. Thus, the informational requirements to obtain the *Nash Equilibrium* might not be fulfilled and therefore we should also consider other equilibrium concepts.

As already mentioned in the introduction, imitation is a plausible and simple decision rule. A decision rule based on imitation would be for example: *"I will set the price of the firm which made the highest profit in the last period"*. The informational requirements for such a decision rule are much lower. Only the profits and the prices have to be known by both firms. Additionally, to intuitively imitate the behavior of the firm with the greatest success appears to be a straight decision rule for success.

Also, as mentioned in Vega-Redondo (1997*b*) "this rule seems intuitively appealing if the decision makers are rewarded according to their relative performance", which is another motivation for the decision maker to imitate[1].

For a symmetric game like our model these arguments are getting even stronger. Therefore, we will formalize imitation behavior in this chapter and subsequently calculate the resulting *Imitation Equilibrium*.

Note that the *Imitation Equilibrium* is a dynamical approach whereas

[1]As we will see later on, *Relative Payoff Maximization* and imitation behavior are closely connected.

the *Nash Equilibrium* concept is a statical one. Thus, we have to extend the model described in section 1.1.

2.1 Extensions to the model

First, we have to introduce periods in which the imitation dynamic will be embedded. In our approach we will use discrete time and periods will be indexed by $t \in \mathbb{N}_0$. In each round both firms set their prices, hence the process has at the period t the state $\omega(t) = (p_1(t), p_2(t))$. Due to technical considerations we assume that both firms can only set prices that are in the finite grid $\Gamma = \{0, \delta, 2\delta, \ldots, \gamma\delta\}$, where $\gamma \in \mathbb{N}$ and $\delta \in \mathbb{R}_{++}$.

With these assumptions on the prices we will have a finite state space in the *Markov chain* which allows us to use a less complicated technical procedure for analyzing the long run behavior. It will additionally be assumed that the resulting imitation equilibrium price is an element of the price grid Γ. However, these assumptions do not establish strong restrictions because the price grid Γ can be arbitrarily fine due to the parameters δ and γ.

With $\pi(t) = (\pi_1(t), \pi_2(t))$ we will denote the profit profile in period t. The profit of player i in round t can be formalized as (see equation 1.4):

$$\pi_i(t) = \frac{1}{2t}(p_i(t) - c)(p_j(t) + t - p_i(t))$$

This profit profile in period t can be directly calculated from the price profile in the same period.

Revision probability Now let us assume that with a certain probability (independent in respect to time and players) $w > 0$ a firm is allowed to set another price in the next round. This probability w is called revision probability.

"Imitation of the best" dynamics If the firm receives a revision opportunity, the firm sets the price of the firm which made the highest profit. In addition we assume that the firm randomly chooses a price (the own price or the price of the other) if both firms have the same profit (*random tie breaking*). The probability distribution over these both prices is irrelevant; it is only important that the probability of setting each of these two prices is strictly positive.

2.2 Process as a Markov chain

The described process can be modeled as a *discrete-time Markov chain*[2] because future state(s) of the process depend only on the current state. In this as well as the next subsection, we will provide a brief and basic, but far from being complete, introduction to *Markov chains*, so that readers without basic knowledge in stochastic processes can follow the analysis.

A Markov chain is a random process that has no memory which means that the next state in the next period only depends on the current state. The state space of our process is given by $\Omega = \Gamma \times \Gamma = \Gamma^2$ and is thus the set of all possible price combinations of the two firms. As we have assumed, a finite price grid of the resulting whole state space is also finite. The transition matrix of a *Markov chain*, denoted for this process by P^0 (we are using the superscript 0 because we want to differentiate this unperturbed process from the perturbed one P^ϵ which will be introduced later), depicts the probabilities to move between two states. Therefore, the element

$$p_{ij} = \mathbb{P}\big(\omega(t+1) = j \mid \omega(t) = i\big)$$

of the matrix P^0 is the probability to move from state i to state j ($i, j \in \Omega$) in one period. The probability to reach the state j from the state i in n periods is denoted by $p_{ij}^{(n)}$. This probability can be calculated by multiplication of the transition matrix. So $p_{ij}^{(n)}$ is the i, j element of the matrix P^n.

Communication classes Every state space of *Markov chains* can be partitioned in communication classes $\mathbb{C}_k, k = 1, 2, \ldots, l$. The condition for states being in the same communication class is that they can *communicate* with each other. *Communicating* in this context means that there is a positive probability to reach one state from the other state and vice versa. In mathematical terms:

$$\forall k, \forall i, j \in \mathbb{C}_k, i \neq j, \exists n, m \in \mathbb{N} \text{ so that } p_{ij}^{(n)} > 0 \text{ and } p_{ji}^{(m)} > 0$$

Closed communication classes A communication class $\mathbb{C}_k$ is closed if there is no way out of it. In other words: If the process enters in a closed communication class, it will remain forever in that class. More formally:

$$\forall i \in \mathbb{C}_k, \forall j \notin \mathbb{C}_k, \forall n \in \mathbb{N} \text{ holds } p_{ij}^{(n)} = 0$$

[2]see Chapter 1 of Norris (1997)

A communication class which consists only of one state is called a singleton. Another definition which we will use in the following proposition are monomorphic states. A monomorphic state is a price profile which has the same price for all firms.

2.3 Unperturbed Process

2.3.1 Closed Communication Classes

Proposition 1. *In the specified model all closed communication classes are singletons and monomorphic states.*

Proof. Clearly, each monomorphic state is absorbing. This is trivial because if both firms have the same profit, the firm(s) which get(s) a revision opportunity will not change its price, as imitation of the other firms' price leads to the same price and thus to the same price profile.

Furthermore, all other singletons are not absorbing: First, if the firms have different profits, the state is clearly not absorbing. In such a state the firm with the lower profit will adapt (with probability w) the price of the other firm. If the firms had exactly the same profit but different prices, [3] this singleton would also not be absorbing as there is a positive probability to leave that state. To leave this state can for example be achieved by a revision opportunity of one firm and the possibility of choosing the price of the other firm (*random tie breaking*).

It is also trivial to see that all closed communication classes have to be singletons. Due to the fact that the Markov chain can reach an absorbing monomorphic state from every non-monomorphic state, a non-monomorphic state can never be part of a closed communication class. It is also clear that an absorbing communication class can never contain more than one monomorphic state, since monomorphic states do not communicate with each other, meaning there is no opportunity to reach one monomorphic state from another monomorphic state. □

Every Markov process will finally end up in a closed communication class. This proposition only ensures that the process in the unperturbed case will end up[4] at one of the monomorphic price profiles. However we are still not

[3] This is possible: For example if one firm sets $p_i = c$ and the other $p_j > c$ the profit for both firms would be zero.

[4] Almost sure, this means that the probability that it will happen is one.

able to favor one of the closed communication classes as *Imitation Equilibrium.*

2.3.2 Path Dependence

The analysis of the unperturbed process could not distinguish between all the monomorphic states because every monomorphic state can be reached[5] and will then be played forever on. The monormophic state in which the process ends up depends strongly on the initial price profile. Therefore, the finally selected state is not independent of the initial price profile. Such a behavior is called path dependent.

In order to get rid of this problem we will introduce small perturbations in each round of the process. The analysis of such a perturbed process is closely connected to the analysis of the unperturbed process, so we can use the results made here, in particular proposition 1, for the analysis of the perturbed process.

2.4 Perturbed Process

The perturbed process can be viewed as an extension of the unperturbed process. In the perturbed process, the imitation dynamics will be enriched with a mutation mechanism. The mutation comes into effect after the imitation. After adjusting their prices in the revision step, firms will mutate with a small independent probability $\epsilon > 0$. If they mutate, they will choose a price in the price grid Γ randomly. The only assumption on the random selection is that every price in the grid is selected with a positive probability ($\forall p \in \Gamma, \forall i \in \{1, 2\}$ holds $\mathbb{P}_i(p) > 0$). So the distribution has be neither uniform nor the same for both players.

This mutation can be interpreted economically in different ways. One possible explanation is that mutations are simply errors of the firms which occur with a small probability. These errors could also be interpreted as experiments of the firm. An economic motivation of such experiments would be that sometimes it happens that a firm will be replaced by a new firm. The newly entered firm does not know which prices were played in the round before. Without any information this firm will experiment setting one of the possible prices at random.

[5] With a high revision probability this monomorphic state will be reached already after few rounds.

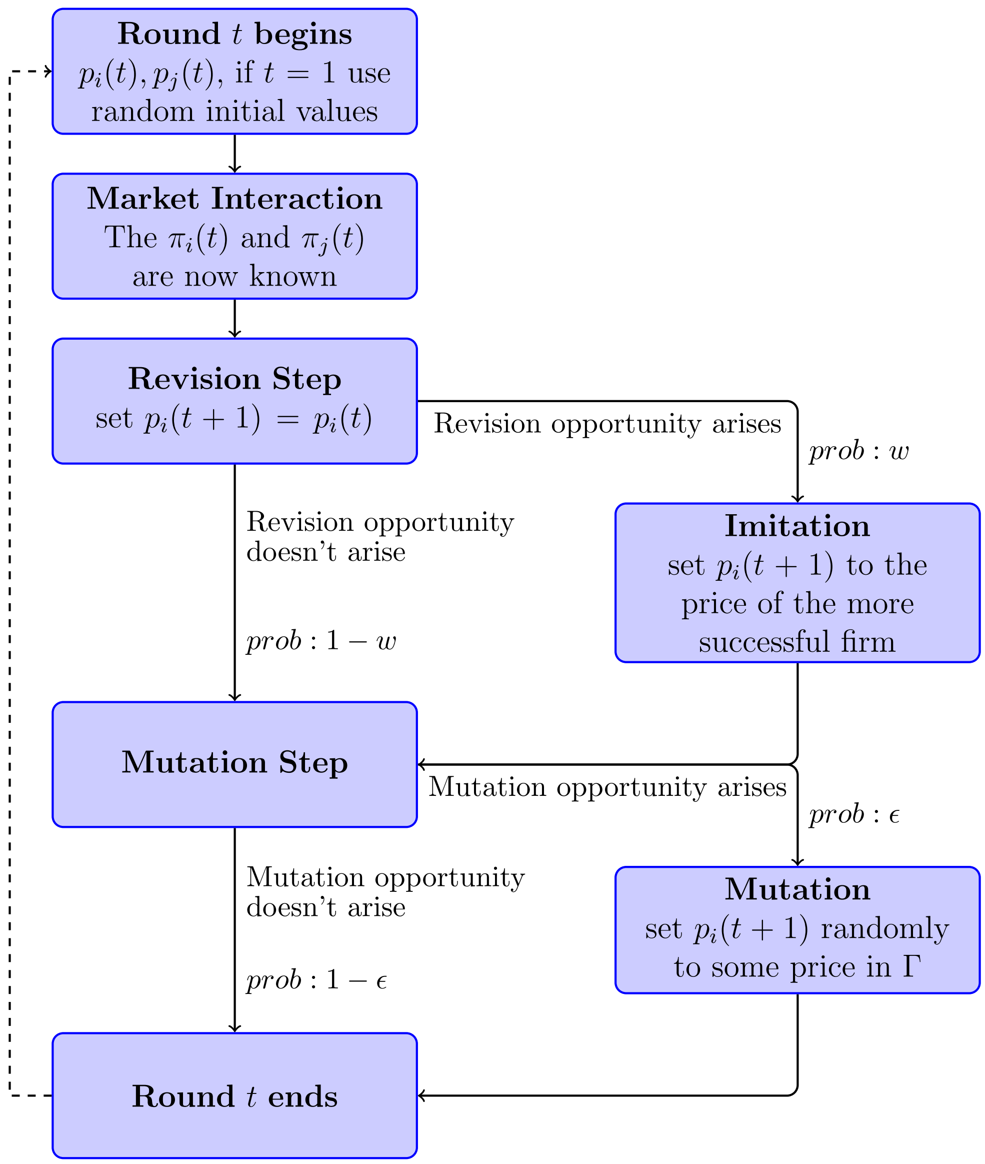


Figure 2.1: The steps in the perturbed model for player i

In the figure 2.1 the flow chart of a round with the revision opportunity and mutation mechanism is shown sequentially. This chart shows only the flow for the firm i but it is the same for player j.

2.4.1 Mathematical Implications

Through the introduction of mutation into the Markov chain the process becomes *irreducible*. Irreducibility means that in the process only one communication class exists which contains all states of the state space Ω. The presence of mistakes makes any transition between two states possible. Thus there is an unique closed communication class which contains all states.

As any state can be reached from any state in the process, this process is also *ergodic*[6]. This property allows us to get rid of the path dependency of the unperturbed process. Every ergodic *Markov chain* has a unique invariant distribution, denoted by μ, over the state space. This means that if the process runs forever ($t \to \infty$), the average fraction of time it stays in every state equals the value given by the invariant distribution. Clearly, this distribution depends on the mutation rate ϵ as well as on the revision probability w. We will focus on the dependency of the invariant distribution on the mutation rate and will therefore write $\mu(\epsilon)$ as a function of the mutation probability.

2.4.2 Solution Concept

As we are assuming positive mutation rates, the calculation of the invariant distribution $\mu(\epsilon)$ for a fixed value of ϵ would be the best characterization of the behavior of the process. This explicit computation of $\mu(\epsilon)$ is mathematically difficult. Since the calculation of

$$\mu^\star = \lim_{\epsilon \to 0} \mu(\epsilon)$$

is easier and still gives a good approximation for very small ϵ, this limit invariant distribution is mostly used in literature.[7]

Usually, the whole weight of the invariant distribution $\mu^\star$ is only concentrated on one or few states by the limiting process. The states with positive weights in the limit of the invariant distribution are called *Long Run Equilibria* (*LRE*) of the process.

[6]This implication holds only in processes with an finite state space.

[7]see Ellison (2000, p. 22)

So if only one state has the whole weight of the distribution and the process runs forever, the process would in average *almost surely* be in that specific state.

There are different approaches to determine the *LRE*. In this work we will focus on the *Freidlin and Wentzell* approach. Another technique is for example the *(modified) Radius-Coradius* approach which was introduced by Ellison (2000).

Freidlin and Wentzell

We will use the same technique as in Vega-Redondo (1997*b*). He used the Freidlin & Wentzell (1984) approach to calculate the long run equilibrium of the *Markov process*. This is a graph theoretic approach, which leads directly to the *LRE*. The first economic works which used this approach to determine the *LRE* have be done by Kandori et al. (1993) and Young (1993). This approach works basically as follows[8]:

For every closed communication class $h' \in \mathbb{H}$ ($\mathbb{H}$ denotes the set of all closed communication classes)[9] of the unperturbed process P^0 we define $\mathbb{S}_{h'}$ as the set of all possible spanning trees over all $h \in \mathbb{H}$ which are rooted in h'. A spanning tree is a directed tree, in which from every vertex a unique path to the root vertex exists. Thus, the spanning tree has no cycles and every vertex, except for the root, has one outgoing edge.

All directed edges (h_i, h_j) of a spanning tree $s \in \mathbb{S}_{h'}$ are assigned the costs of a move from the communication class h_i to h_j in the process P^ϵ. The costs of moving are simply the required numbers of mutations to reach the communication class h_j from h_i. At this point, we let the costs of a the whole spanning tree s be denoted by $c(s)$, which is just the sum of all edges of the spanning tree s.

The stochastic potential σ_h of a communication class h is given by the cost of the spanning tree that has minimal costs. Therefore we have:

$$\sigma_{h'} = \min_{s \in \mathbb{S}_{h'}} c(s)$$

Finally, we only have to compare the stochastic potential of all communication classes. The communication classes with the minimal stochastic

[8]A more detailed introduction can be found for example in Vega-Redondo (1997*a*, Chapter 5)

[9]As shown in our case before, the set of all closed communication classes are all monomorphic singletons.

potential are the *Long Run Equilibria*.

Based on the LRE we define the *Imitation Equilibria*.

Definition 1. ***Imitation Equilibria*** *We call the LRE of the 'imitation of the best' dynamics 'Imitation Equilibria'.*

Note that all *Long Run Equilibria* of a Markov process, which uses the described imitation dynamics, are monomorphic and singletons.[10] Therefore, we can refer with *Imitation Equilibria* to states and not to communication classes. We will denote the price used in the price profile of the monomorphic *Imitation Equilibrium* by *Imitation Price*.

2.4.3 Analysis

In this section we apply the *Freidlin and Wentzell* approach to our model.Let us begin with the question: *Which small mutation steps on the price grid will be followed?*

This question can be mathematically formalized in the following inequality: $\pi_i(p+\delta,p) \geq \pi_j(p+\delta,p)$. First, let us only consider upward mutations[11].

$$\begin{aligned}
\pi_i(p+\delta,p) &\geq \pi_j(p+\delta,p) \\
\pi_i(p+\delta,p) &\geq \pi_i(p,p+\delta) \\
(p+\delta-c)s_i(p+\delta,p) &\geq (p-c)s_i(p,p+\delta) \\
(p+\delta-c)\frac{1}{2t}(t-\delta) &\geq (p-c)\frac{1}{2t}(\delta+t) \\
\delta t-\delta^2+2c\delta &\geq 2p\delta \\
p &\leq c+\frac{t}{2}-\frac{\delta}{2}
\end{aligned}$$

As the resulting upward mutation prices are elements of the price grid Γ, the price $c+\frac{t}{2}-\delta$ is the highest price that fulfills this inequality.

The same analysis can be done for downward mutations:

[10] See proposition 1.

[11] We are assuming here that δ is sufficiently small, at least smaller than the transport costs t so that both firms have a market share greater than zero.

$$\begin{aligned}
\pi_i(p-\delta,p) &\geq \pi_j(p-\delta,p)\\
\pi_i(p-\delta,p) &\geq \pi_i(p,p-\delta)\\
(p-\delta-c)s_i(p-\delta,p) &\geq (p-c)s_i(p,p-\delta)\\
(p-\delta-c)\frac{1}{2t}(t+\delta) &\geq (p-c)\frac{1}{2t}(t-\delta)\\
2p\delta &\geq 2c\delta+\delta t+\delta^2\\
p &\geq c+\frac{t}{2}+\frac{\delta}{2}
\end{aligned}$$

Here we get the same result. Thus downward mutations will be followed until the price $c+\frac{t}{2}$ is reached. We anticipate that the price $p^\star = c+\frac{t}{2}$ will be our *Imitation Equilibrium*, therefore this price will be in the price grid Γ by assumption. This means that all 'one step' upward and downward mutations will be followed until this price is reached.

We have found a candidate for the *Imitation Equilibrium*. Note that the obtained result ensures only that 'one step' upward and downward mutations will not permit the abandonment of the price profile $\omega^\star := (p^\star, p^\star)$. It could still be possible that other types of mutations allow the abandonment of this state. To ensure that this is not possible we prove the following lemma:

Lemma 2. *A firm which uniquely sets the price $p^\star$ has a profit higher than the profit of the other firm.*

Proof. We claim:

$$\pi_i(p_i,p_j) \geq \pi_j(p_i,p_j) \qquad (2.1)$$
$$\text{for } p_i = c+\frac{1}{2}t$$

We obtain by using 1.4:

$$\big((c+\frac{1}{2}t)-c\big)\big(p_j+t-(c+\frac{1}{2}t)\big) \geq \big(p_j-c\big)\big((c+\frac{1}{2}t)+t-p_j\big)$$

Simplification of this term leads to:

$$p_j^2-2p_jc-tp_j+c^2+tc+\frac{1}{4}t^2 \geq 0$$

By transforming[12] this expression we obtain:

$$p_j^2-2p_j(c+\frac{1}{2}t)+(c+\frac{1}{2}t)^2 \geq 0$$

[12] The equality $c^2+tc+\frac{1}{4}t^2=(c+\frac{1}{2}t)^2$ is used here.

Finally we obtain a quadratic function which is clearly greater than zero:

$$\big(p_j - (c + \frac{1}{2}t)\big)^2 \geq 0$$

and therefore:

$$\big(p_j - (c + \frac{1}{2}t)\big)^2 > 0 \qquad \text{if } p_j \neq c + \frac{1}{2}t$$

□

With this lemma we can apply the *Freidlin and Wentzell* approach to this Markov process.

To switch from one state in a closed communication class to a state in another communication class at least one mutation is needed. The analysis above has shown us that this price leads in the worst case to the profit of the other firm or otherwise to a higher profit. Therefore, the closed communication class which contains the state $\omega^\star := (c + \frac{1}{2}t, c + \frac{1}{2}t)$, denoted by $h_{\omega^\star}$, needs two mutations to be abandoned.

On the contrary, to move from any other closed communication class to $h_{\omega^\star}$, only one mutation is required. The lemma 2 ensures that in this case the profit of the mutating firm is higher than the profit of the non-mutating firm.

According to these characteristics of the required mutations, we can construct a spanning tree which is rooted in the closed communication class $h_{\omega^\star}$ with a stochastic potential of:

$$\sigma_{h_{\omega^\star}} = \min_{s \in \mathbb{S}_{h_{\omega^\star}}} c(s) = |\Gamma| - 1 = \gamma$$

$|\Gamma|$ is the cardinality of the set Γ. The tree has $|\Gamma| = \gamma + 1$ nodes and, therefore, $|\Gamma| - 1 = \gamma$ edges. A possible way to construct such a spanning tree would be to simply connect every closed communication class with the root (see figure 2.2).

Now we only have to show that all other possible spanning trees, with a different communication class as root, have a higher stochastic potential.

We know that every spanning tree with a different root has to contain an outgoing edge from the communication class $h_{\omega^\star}$. This edge has the cost of two mutations. Also every other directed edge has at least the cost of one mutation, so we get:

$$\sigma_{h_\omega} = \min_{s \in \mathbb{S}_{h_\omega}} c(s) \geq |\Gamma| = \gamma + 1 \text{ for } \forall \omega \in \mathbb{H}, \omega \neq \omega^\star$$

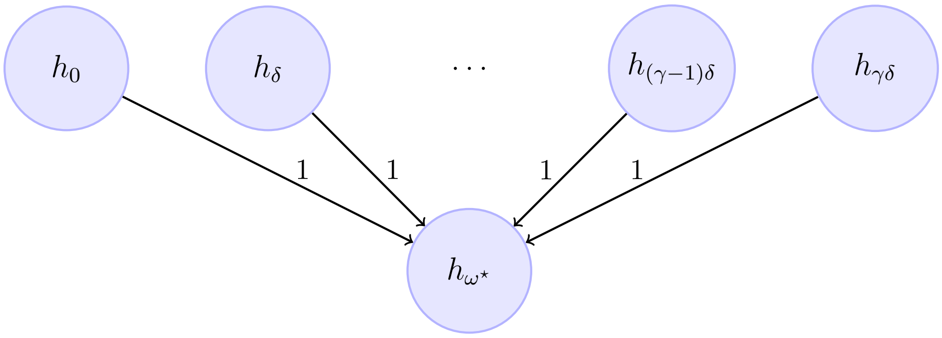


Figure 2.2: Spanning tree with $h_{\omega^\star}$ as root

Hence, we get the inequality:

$$\sigma_{h_{\omega^\star}} < \sigma_{h_\omega} \text{ for } \forall \omega \in \mathbb{H}, \omega \neq \omega^\star$$

Accordingly, the closed communication class $h_{\omega^\star}$ is the class with the lowest stochastic potential and therefore the long run equilibrium communication class. The *Imitation Equilibrium* is therefore the price profile $(p^\star, p^\star)$ with $p^\star = c + \frac{1}{2}t$.

Relative Payoff Maximization

At this point we will add another interesting derivation of the *Imitation Equilibrium*. Alós-Ferrer & Ania (2005) have shown (proposition 4 of the paper) that a strictly, globally stable finite *Evolutionary Stable Set* (ESS) is also the unique stochastically stable state with the imitation dynamics. The intuition behind this is quite simple: ESS can be defined as relative payoff maximizer[13]. Thus playing an ESS does not maximize the own profit, it tries instead to perform relatively better than the others[14]. Exactly this relative performance is the basic principle of imitation.

The definition of an ESS was originally introduced by Maynard Smith & Price (1973). As this definition captured only infinite population games, Schaffer (1988) introduced a formal definition for finite population games. Applied to our model this definition leads to:

Definition 2. *$p^\star$ is a finite ESS if for all $p \in \Gamma$:*

$$\pi(p^\star, p) \geq \pi(p, p^\star)$$

[13] An ESS is a *Nash Equilibrium* in the derived game with relative payoffs.

[14] Such an behavior is called 'spiteful behavior', see Alós-Ferrer & Ania (2005, page 9).

And $p^\star$ is a strict finite ESS if the equality is only achieved for $p = p^\star$.

Note that in a game with the population size of two (like in the *Hotelling Game*) every (strict) finite ESS is also a globally stable (strict) finite ESS. This is simply due to the fact that in a two-player game, at most one player can mutate and still play against a player which plays the ESS strategy.

With Lemma 2, we have already shown that the strategy price $p^\star$ is a strictly, globally stable ESS and therefore an *Imitation Equilibrium*. However, we will provide another proof since we will use this approach later on to directly determine the *Imitation Equilibrium*.

An equivalent definition for the ESS in symmetric aggregative games is that the ESS are maximizing the relative payoffs. Hence, the *Imitation Equilibrium* can also be derived as follows:

$$\operatorname*{argmax}_{p_i} \pi_i(p_i, p_j) - \pi_j(p_i, p_j) =$$

We get the following expression by using the profit formula 1.4:

$$\operatorname*{argmax}_{p_i} \frac{1}{2t}\Big((p_i - c)(p_j + t - p_i) - (p_j - c)(p_i + t - p_j)\Big)$$

Solving this maximization problem, we calculate the first derivative:

$$\frac{d(\pi_i(p_i, p_j) - \pi_j(p_i, p_j))}{dp_i} = \frac{1}{2t}(t - 2p_i + 2c)$$

By setting this derivative to zero we obtain the price p_i which maximizes[15] the relative profit of firm i:

$$0 = \frac{1}{2t}(t - 2p_i + 2c) \implies p_i = c + \frac{1}{2}t \tag{2.2}$$

By applying the proposition 4[16] of Alós-Ferrer & Ania (2005) the proof that the state $(c+\frac{1}{2}t, c+\frac{1}{2}t)$ is the unique *Imitation Equilibrium* is completed.

[15] The second derivation leads to $-\frac{2}{t}$ which is smaller than zero, therefore, we obtain a maximum.

[16] "Let $\Gamma \equiv (N, S, \Pi)$ be a symmetric N-player game with finite S. Let $s^\star$ be a strictly globally stable ESS. Then, the profile $(s^\star, \ldots, s^\star)$ is the unique stochastically stable state of the imitation dynamics with experimentation."

2.5 Remarks

Consequently, the obtained *Imitation Price* is lower than the *Nash Equilibrium* price (see 1.6) but still above the marginal cost pricing. Interestingly enough, the *Imitation Price* lies exactly between marginal cost pricing and the *Nash Equilibrium* price.

Prices are lower and therefore also the profits of the firms are lower; to be more precise: the profit of the firms in the *Imitation Equilibrium* are only half of the *Nash Equilibrium* profits.

Chapter 3

Modifications

In this chapter we will drop some assumptions which we have made before for the simple model introduced in section 1.1. It will be shown that some of the used assumptions are essential to obtain a unique *Imitation Equilibrium*.

3.1 Distribution of the consumers

At the beginning, as the first modification, we will start modifying the distribution of the consumers in the unit interval. Until now we have simply used a uniform distribution (see figure 1.1).

3.1.1 Distribution

We will construct a quite simple distribution that allows us to center the weight of the consumer distribution on the middle of the unit interval with a parameter α.

By simply using the property that the function $f(x) := x^\alpha$ has for all values of $\alpha \in \mathbb{R}_{++}$ the function values $f(0) = 0$ and $f(1) = 1$ and the property that the function *monotonically increases*[1] in different shapes depending on the parameter α it is quite simple to construct such a distribution function. We will only consider symmetric distributions.

By scaling, copying and mirroring the interval $[0, 1]$ of the function we

[1]We only consider the interval $[0, 1]$.

can construct the following cumulative distribution function:

$$F_\alpha(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2}(2x)^\alpha & 0 \leq x < \frac{1}{2} \\ 1 - \frac{1}{2}\big(2(1-x)\big)^\alpha & \frac{1}{2} \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

Derivation leads to the density function:

$$f_\alpha(x) = \begin{cases} \alpha(2x)^{\alpha-1} & 0 \leq x < \frac{1}{2} \\ \alpha\big(2(1-x)\big)^{\alpha-1} & \frac{1}{2} \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

Limit Cases

The upper limit case $\lim_{\alpha\to\infty} f_\alpha(x)$ leads to a distribution that has the entire weight at one point. This point lise exactly at the middle of the unit interval[2].

The lower limit case $\lim_{\alpha\to 0\downarrow} f_\alpha(x)$ puts the half of the entire weight on each of the two ends of the unit interval.

By setting $\alpha = 1$ we obtain the uniform distribution, so the uniform distribution used in the simple model is a special case of this parametric distribution function. Figure 3.1 illustrates the density and cumulative distribution function for some different values of α. There it can be seen that a value greater than one leads to a higher weight of consumers in the center area of the unit interval, whereas values below one cause the contrary.

3.1.2 Profit and market share

The market share can be analogously calculated like in the simple model. The only difference is that the obtained market share of the uniform case has to be weighted by the cumulative distribution function.

So the market share of player i given the prices p_i and p_j denoted by $s_i^\alpha(p_i, p_j)$ can be calculated with $s_i^\alpha(p_i, p_j) = F_\alpha\big(s_i(p_i, p_j)\big)$.

The profit function can be directly derived from (1.3):

[2]Can also be seen by the variance of the probability distribution: $\lim_{\alpha\to\infty} \mathrm{Var}(X_\alpha) = \lim_{\alpha\to\infty} \frac{1}{2(1+\alpha)(2+\alpha)} = 0$

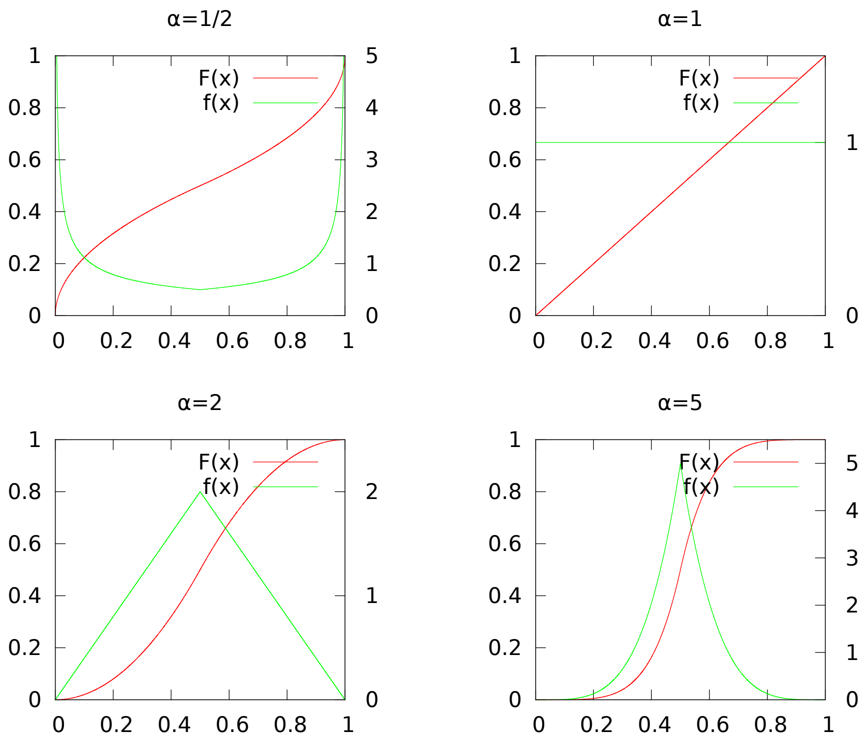


Figure 3.1: Density and cumulative distribution function for different values of α

$$\pi_i^\alpha(p_i,p_j) = (p_i-c)s_i^\alpha(p_i,p_j) = (p_i-c)F_\alpha\big(s_i(p_i,p_j)\big)$$

3.1.3 Nash Equilibrium

Proposition 3. *For $\alpha \in [1,\infty)$ a* Nash Equilibrium *is given by:*

$$p^\star_{i/j} = c + \frac{t}{\alpha} \tag{3.1}$$

The proof of this proposition can be found in the in the appendix A.1.

3.1.4 Imitation Equilibrium

Here we will use the same procedure that we have already used in the analysis of the simple model (section 2.4.3).

First we are looking which small upward price mutation will be followed by the other firm.

$$\begin{aligned}
\pi_i^\alpha(p+\delta,p) &\geq \pi_j^\alpha(p+\delta,p)\\
\pi_i^\alpha(p+\delta,p) &\geq \pi_i^\alpha(p,p+\delta)\\
(p+\delta-c)s_i^\alpha(p+\delta,p) &\geq (p-c)s_i^\alpha(p,p+\delta)\\
(p+\delta-c)F_\alpha\big(s_i(p+\delta,p)\big) &\geq (p-c)F_\alpha\big(s_i(p,p+\delta)\big)\\
(p+\delta-c)F_\alpha\big(s_i(p+\delta,p)\big) &\geq (p-c)\Big(1-F_\alpha\big(s_i(p+\delta,p)\big)\Big)\\
p-c &\leq (2p+\delta-2c)F_\alpha\Big(\underbrace{s_i(p+\delta,p)}_{<\frac{1}{2},\text{ because }\delta>0}\Big)\\
p-c &\leq \frac{1}{2}(2p+\delta-2c)\big(2s_i(p+\delta,p)\big)^\alpha\\
p-c &\leq \frac{1}{2}(2p+\delta-2c)\Big(2\big(\frac{p+t-(p+\delta)}{2t}\big)\Big)^\alpha\\
p-c &\leq \frac{1}{2}(2p+\delta-2c)\Big(\frac{t-\delta}{t}\Big)^\alpha\\
p\Big(1+\big(\frac{t-\delta}{t}\big)^\alpha\Big) &\leq \frac{1}{2}(\delta-2c)\Big(\frac{t-\delta}{t}\Big)^\alpha + c\\
p &\leq \frac{\frac{1}{2}(\delta-2c)\big(\frac{t-\delta}{t}\big)^\alpha + c}{1+\big(\frac{t-\delta}{t}\big)^\alpha}
\end{aligned}$$

To get the exact limit where no upward price mutation would be followed independently of δ, we calculate the limit of this obtained expression as δ approaches 0. This can be seen as infinite refinement of the price grid.

$$\lim_{\delta\to 0}\frac{\frac{1}{2}(\delta-2c)\left(\frac{t-\delta}{t}\right)^{\alpha}+c}{1+\left(\frac{t-\delta}{t}\right)^{\alpha}}$$

This expression leads to the indeterminate form $\frac{0}{0}$. Due to *L'Hôpital's rule*[3] the limit can be calculated by limiting the derivation of the numerator and denominator:

$$\lim_{\delta\to 0}\frac{\frac{1}{2}(\delta-2c)\left(\frac{t-\delta}{t}\right)^{\alpha}+c}{1+\left(\frac{t-\delta}{t}\right)^{\alpha}}=\lim_{\delta\to 0}\frac{\frac{1}{2}+\frac{\alpha c}{t}}{\frac{\alpha}{t}}=c+\frac{t}{2\alpha}$$

The same calculation can also be done for downward mutation:

$$\begin{aligned}
\pi_i^{\alpha}(p-\delta,p) &\geq \pi_j^{\alpha}(p-\delta,p)\\
\pi_i^{\alpha}(p-\delta,p) &\geq \pi_i^{\alpha}(p,p-\delta)\\
(p-\delta-c)s_i^{\alpha}(p-\delta,p) &\geq (p-c)s_i^{\alpha}(p,p-\delta)\\
(p-\delta-c)F_{\alpha}\big(s_i(p-\delta,p)\big) &\geq (p-c)F_{\alpha}\big(s_i(p,p-\delta)\big)\\
(p-\delta-c)\Big(1-F_{\alpha}\big(\underbrace{s_i(p,p-\delta)}_{<\frac{1}{2}}\big)\Big) &\geq (p-c)F_{\alpha}\big(s_i(p,p-\delta)\big)\\
p-\delta-c &\geq \frac{1}{2}(2p-\delta-2c)\Big(\frac{t-\delta}{t}\Big)^{\alpha}\\
p\Big(1-\big(\frac{t-\delta}{t}\big)^{\alpha}\Big) &\geq -\frac{1}{2}(\delta+2c)\Big(\frac{t-\delta}{t}\Big)^{\alpha}+\delta+c\\
p &\geq \frac{-\frac{1}{2}(\delta+2c)\left(\frac{t-\delta}{t}\right)^{\alpha}+\delta+c}{1-\left(\frac{t-\delta}{t}\right)^{\alpha}}
\end{aligned}$$

By calculating the limit by using *L'Hôpital's rule* we obtain:

$$\lim_{\delta\to 0}\frac{-\frac{1}{2}(\delta+2c)\left(\frac{t-\delta}{t}\right)^{\alpha}+\delta+c}{1-\left(\frac{t-\delta}{t}\right)^{\alpha}}=\lim_{\delta\to 0}\frac{\frac{1}{2}+\frac{\alpha c}{t}}{\frac{\alpha}{t}}=c+\frac{t}{2\alpha}$$

The upward and the downward mutation limits are the same, therefore we obtain a price candidate for the *Imitation Equilibrium*. We will use again the

[3] also known as Bernoulli's rule

proposition 4 of Alós-Ferrer & Ania (2005) and prove that the obtained candidate is the unique *Imitation Equilibrium* by proving that the price strategy $c + \frac{t}{2\alpha}$ is an *Evolutionary Stable Strategy.*

The proof only captures the parameter values $\alpha \geq 1$ and it is important to mention that it does not hold for $\alpha < \frac{1}{2}$.

Proposition 4. *The price $c + \frac{t}{2\alpha}$ is the Imitation Equilibrium price for all $\alpha \geq 1$.*

Proof. We will prove this proposition by showing that the price $p = c + \frac{t}{2\alpha}$ is an globally stable strict finite ESS.

To do that we have to show that[4]:

$$\forall \alpha \geq 1, \forall p \in \Gamma, p \neq c + \frac{t}{2\alpha} : \pi_i^\alpha\big(c + \frac{t}{2\alpha}, p\big) > \pi_j^\alpha\big(c + \frac{t}{2\alpha}, p\big)$$

To simplify matters we will set the marginal costs in this proof to zero. The following result however holds also for positive marginal costs.

$$\begin{aligned}
\pi_i^\alpha\big(\frac{t}{2\alpha}, p\big) &> \pi_j^\alpha\big(\frac{t}{2\alpha}, p\big)\\
\pi_i^\alpha\big(\frac{t}{2\alpha}, p\big) &> \pi_i^\alpha\big(p, \frac{t}{2\alpha}\big)\\
\frac{t}{2\alpha} s_i^\alpha\big(\frac{t}{2\alpha}, p\big) &> p\, s_i^\alpha\big(p, \frac{t}{2\alpha}\big)
\end{aligned}$$

Because of the piecewise definition of the distribution function we have to consider the two different cases $0 \leq p < \frac{t}{2\alpha}$ and $p > \frac{t}{2\alpha}$. Let us start with the first case:

$$\begin{aligned}
\frac{t}{2\alpha} F_\alpha\Big(s_i\big(\frac{t}{2\alpha}, p\big)\Big) &> p\Bigg(1 - F_\alpha\Big(s_i\big(\frac{t}{2\alpha}, p\big)\Big)\Bigg)\\
\Big(\frac{t}{2\alpha} + p\Big) F_\alpha\Big(\underbrace{s_i\big(\frac{t}{2\alpha}, p\big)}_{< \frac{1}{2}}\Big) &> p\\
\frac{1}{2}\Big(\frac{t}{2\alpha} + p\Big)\Big(2\big(\frac{p - t - \frac{t}{2\alpha}}{2t}\big)\Big)^\alpha &> p\\
\frac{1}{2}\Big(\frac{1}{2\alpha} + \frac{p}{t}\Big)\Big(1 - \frac{1}{2\alpha} + \frac{p}{t}\Big)^\alpha &> \frac{p}{t}
\end{aligned}$$

[4] $\pi_i^\alpha\big(c + \frac{t}{2\alpha}, p\big) = \pi_j^\alpha\big(c + \frac{t}{2\alpha}, p\big)$ for $p = c + \frac{t}{2\alpha}$ trivially holds.

Now we will substitute $\frac{p}{t}$ with x. The considered interval is therefore: $0 \leq x < \frac{1}{2\alpha}$

$$f(x) := \frac{1}{2}\Big(\frac{1}{2\alpha} + x\Big)\Big(1 - \frac{1}{2\alpha} + x\Big)^{\alpha} - x > 0$$

Now we will prove that this function (denoted by $f(x)$) is strictly positive in the interval $[0, \frac{1}{2\alpha})$ for the parameter value $\alpha \geq 1$.

Derivation of the function $f(x)$ leads to:

$$\frac{df(x)}{dx} = \frac{1}{2}\Bigg(\underbrace{\alpha\Big(1 - \frac{1}{2\alpha} + x\Big)^{\alpha-1}\Big(\frac{1}{2\alpha} + x\Big)}_{<1} + \underbrace{\Big(1 - \frac{1}{2\alpha} + x\Big)^{\alpha}}_{<1}\Bigg) - 1$$

$$\frac{df(x)}{dx} < 0 \text{ for } \forall x \in [0, \frac{1}{2\alpha})$$

So the function is strictly decreasing in the interval $[0, \frac{1}{2\alpha})$. Observing that the function value at $\frac{1}{2\alpha}$ is zero ensures that the inequality $\pi_i^{\alpha}(\frac{t}{2\alpha}, p) \geq \pi_j^{\alpha}(\frac{t}{2\alpha}, p)$ holds for $0 \leq p < \frac{t}{2\alpha}$.

Now we will continue the proof for the other case: $p > \frac{t}{2\alpha}$. We can split this area into the two intervals $I_1 : (\frac{t}{2\alpha}, \frac{t}{2\alpha} + t]$ and $I_2 : (\frac{t}{2\alpha} + t, \infty)$.

The proof for the interval I_2 is trivial, because the difference between the two prices is already large enough for firm i with the higher price to face zero demand and therefore has no market share and no profits. The other firm will have positive profits because the price $p = \frac{t}{2\alpha}$ is above the marginal costs of 0.

For the interval I_1 we get:

$$\begin{aligned}
\frac{t}{2\alpha}\Bigg(1 - F_{\alpha}\Big(s_i(p, \frac{t}{2\alpha})\Big)\Bigg) &> pF_{\alpha}\Big(s_i(p, \frac{t}{2\alpha})\Big) \\
\frac{t}{2\alpha} &> \Big(\frac{t}{2\alpha} + p\Big)F_{\alpha}\Big(\underbrace{s_i(p, \frac{t}{2\alpha})}_{<\frac{1}{2}}\Big) \\
\frac{1}{\alpha} &> \Big(\frac{1}{2\alpha} + \frac{p}{t}\Big)\Big(2(\frac{\frac{t}{2\alpha} - t - p}{2t})\Big)^{\alpha} \\
0 &> \Big(\frac{1}{2\alpha} + \frac{p}{t}\Big)\Big(1 + \frac{1}{2\alpha} - \frac{p}{t}\Big)^{\alpha} - \frac{1}{\alpha}
\end{aligned}$$

Substituting $\frac{p}{t}$ by x, defining the function $g(x)$ for the right-hand side of the inequality and calculating the first derivative results in:

$$\begin{aligned}
\frac{dg(x)}{dx} &= \Big(1+\frac{1}{2\alpha}-x\Big)^{\alpha} - \alpha\Big(1+\frac{1}{2\alpha}-x\Big)^{\alpha-1}\Big(\frac{1}{2\alpha}+x\Big) \\
&= \underbrace{\Big(1+\frac{1}{2\alpha}-x\Big)^{\alpha-1}}_{>0}\underbrace{(\alpha+1)}_{>0}\underbrace{(\frac{1}{2\alpha}-x)}_{<0} \\
\frac{dg(x)}{dx} &< 0 \text{ for } \forall x \in (\frac{1}{2\alpha}, \frac{1}{2\alpha}+1]
\end{aligned}$$

Knowing that $g(\frac{1}{2\alpha}) = 0$ and that the function is decreasing in the interval I_1 we have also proven that the inequality $\pi_i^{\alpha}(\frac{t}{2\alpha}, p) \geq \pi_j^{\alpha}(\frac{t}{2\alpha}, p)$ holds for $p > \frac{t}{2\alpha}$. $\square$

In the appendix B you can find simulation results for α-values greater and lower than 1.

3.1.5 Remarks

As we can see here the result obtained in chapter 2 is also included here.[5]

Another interesting detail of the result is:

$$\lim_{\alpha\to\infty} c + \frac{t}{2\alpha} = c$$

As we can see there is a smooth transition to marginal cost pricing. This does not only hold for the *Imitation Equilibrium*, also the *Nash Equilibrium* converges to marginal cost pricing as $\alpha \to \infty$. So the parameter α with values in the interval $[1, \infty)$ allows a smooth transition from the *Bertrand Competition* to the *Hotelling Game*.

3.2 Transport Costs

A possible modification could be also the the functional form of the transport costs. We will not do this here but we will consider the case of no transport costs at all in this section we. The *Nash Equilibrium* without transport costs is the well known *Bertrand Nash Equilibrium* implying marginal cost pricing as *Nash Equilibrium* in our simple model.

[5] By setting α to 1 we obtain the same result as in the subsection 2.4.3.

Intuitively one would assume that also in the case of zero transport costs the price profile (c, c) would be the only *Imitation Equilibrium* of the Markov process. Yet this does not hold, because for every closed communication class $h_{p'}$ with $p' \geq c$ can be found a spanning tree with the stochastic potential of γ. We will show this in the proof of following proposition.

Proposition 5. *Without transport costs every monomorphic state with a price higher than the marginal costs is a Imitation Equilibrium.*

Proof. If one firm mutates to marginal cost pricing both firms would have zero profits[6], so there is a positive probability that the firm with marginal cost pricing will be imitated. Therefore it is only one mutation needed to reach the communication class h_c which consists of the singleton (c, c). If later on in that state one of the firms would mutate to a higher price $p\prime$ the firm with marginal cost pricing will still be confronted with zero profits. Therefore in the next rounds there exists a positive probability that the price p' would be imitated by this firm. Thus every closed communication class $h_{p'}$ with $p' \geq c$ can be reached with only one mutation from the class h_c. So there exists for every closed communication class $h_{p'}$ with $p' \geq c$ a spanning tree with the stochastic potential of γ. Such a spanning tree is depicted in figure 3.2.

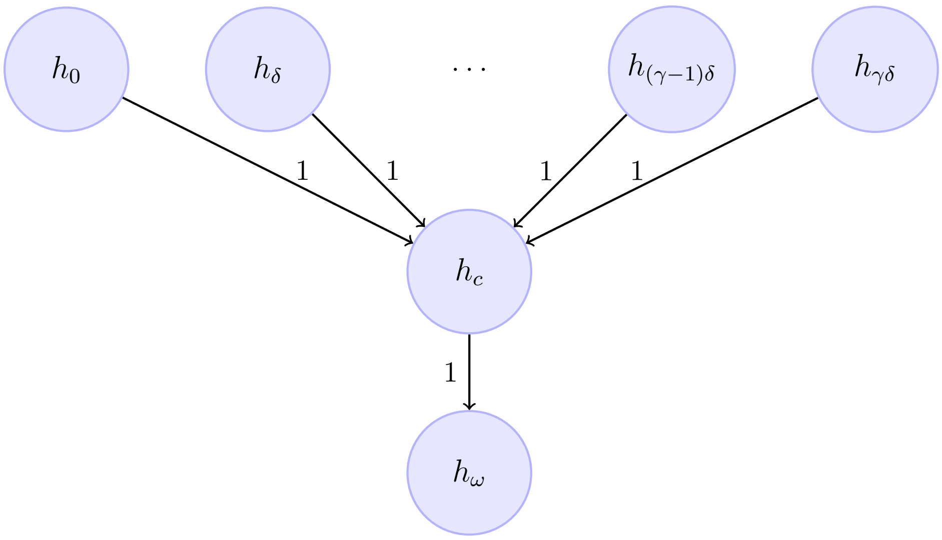


Figure 3.2: Spanning trees ($c_t = 0$)

[6]If the other firm charged a price less than the marginal costs the firm would have negative profits, but than also the following imitation could occur.

Every closed communication class $h_{p'}$ has the same stochastic potential hence all of them are *LRE*. Therefore all monomorphic states are *Imitation Equilibria*.

$\square$

In the appendix B you can find a simulation result of such a process.

This behavior can only occur if c is in the price grid Γ. If this is not the case, the communication class with the smallest price greater than the marginal costs would be the *LRE*.

Imitation in the *Bertrand Competition* has been studied[7] by Alos-Ferrer, Ania & Schenk-Hoppe (2000). Instead of the here resulting trivial *Bertrand Model* they used a n-player game as base model and assume convex production costs.

3.3 Production Costs

3.3.1 Marginal Costs

We already captured symmetric marginal costs in the simple model. By introducing asymmetric marginal costs (denoted by c_i, c_j for the firms i, j respectively) in the *Hotelling Game* we will in general not obtain a unique *Imitation Equilibrium*.[8] If we use *Relative Payoff Maximization* as we did in subsection 2.4.3 we obtain in fact a single maximizer $p_i = \frac{1}{2}(c_i + c_j + t)$, counterpart to equation 2.2, but still this maximizer leads to a negative relative payoff for the firm with the higher marginal costs. Let us denote this relative payoff maximizer with p_{rm} for the following analysis.

To get a better idea of what is happening in such circumstances we will again ask following question: "Which mutation will be followed?" The following inequalities are representing this question mathematically:

[7]They used the same evolutionary approach like we are using in this paper.

[8]However there exists a unique *Nash Equilibrium*: $p_i = \frac{1}{3}(2c_i + c_j) + t$

Upward mutation:

$$\begin{aligned}\pi_i(p+\delta,p) &\geq \pi_j(p+\delta,p)\\(p+\delta-c_i)(t-\delta) &\geq (p-c_j)(t+\delta)\\c_i\delta+c_j\delta+t\delta+c_jt-c_it-\delta^2 &\geq 2p\delta\\p_{rm}+\frac{t}{2\delta}(c_j-c_i)-\frac{\delta}{2} &\geq p\end{aligned}$$

Downward mutation:

$$\begin{aligned}\pi_i(p-\delta,p) &\geq \pi_j(p-\delta,p)\\(p-\delta-c_i)(t+\delta) &\geq (p-c_j)(t-\delta)\\2p\delta &\geq c_i\delta+c_j\delta+t\delta+c_it-c_jt+\delta^2\\p &\geq p_{rm}-\frac{t}{2\delta}(c_j-c_i)+\frac{\delta}{2}\end{aligned}$$

Let us assume that the firm i is the low cost firm, so $c_i < c_j$ holds. These inequalities imply that every upward and downward mutation of firm i away from a price $p \in \Gamma$ and $p \in I$,

$$I := \Big[p_{rm} - \big(\frac{t}{2\delta}(c_j-c_i)-\frac{\delta}{2}\big), p_{rm} + \big(\frac{t}{2\delta}(c_j-c_i)-\frac{\delta}{2}\big)\Big] \tag{3.2}$$

will be followed by the other firm. So if a revision possibility arises only one mutation is needed to reach these states.

This means that for all closed communication classes $h_p, p \in \Gamma \wedge p \in I$ there is a spanning tree with a stochastic potential of γ. All of these communication classes are therefore *LRE* and thus we obtain various *Imitation Equilibria* states.

Now lets look at the expression $\frac{t}{2\delta}(c_j - c_i)$ which is contained in both inequalities. This expression can be arbitrarily large for small δ, meaning that all monomorphic states could be *LRE* if the grid step size is small enough.

This result coincides with the result of Apesteguia, Huck, Oechssler & Weidenholzer (2010). The authors analyzed the *Imitation Equilibrium* in the *Cournot Competition* and figured out that slight differences in the cost structure of the players also leads to multiple *LRE*.

In the appendix B you can find a simulation result of a process with asymmetric marginal costs.

3.3.2 Fixed Costs

Let us denote the fixed costs with f_i, f_j for the firms i, j respectively. Clearly, the *Imitation Equilibrium* will not be influenced by symmetric fixed costs.

Like in the case of asymmetric marginal costs (3.3.1) we can calculate an interval I in which downward and upward mutations of the lower cost firm will be followed by the other firm:

$$I := \Big[p_{im} + \big(\frac{t}{\delta}(f_j - f_i) - \frac{\delta}{2}\big), p_{im} - \big(\frac{t}{\delta}(f_j - f_i) - \frac{\delta}{2}\big)\Big] \tag{3.3}$$

The term p_{im} denotes the *Imitation Equilibrium* in the case of symmetric costs, thus $p_{im} = c + \frac{t}{2}$. This interval is almost the same like the interval I in the case of asymmetric marginal costs (3.2). Therefore also in the case of asymmetric fixed costs the amount of *LRE* depends on the grid step size δ. With a small enough δ all monomorphic states are getting *LRE*.

Comparing the intervals defined in (3.2) and (3.3) shows us that the length of the interval I grows twice as fast in respect to the difference of the fixed cost than to the difference in the marginal cost. This means that the *LRE* is more sensitive to differences in the fixed costs.

3.3.3 Remarks

These results clarify that an imitation behavior is only reasonable in symmetric environments.

If one player has a cost advantage, the other players can never achieve the same success by imitation. On the contrary, the disadvantaged player will also follow ‘stupid’ actions as long as the success of the other player is higher. Exactly this is happening in the previous analyses of the *Imitation Equilibria* in asymmetric environments.

Chapter 4

Simulation

We already mentioned in section 2.4.2 that the limit invariant distribution $\mu^\star$ is used because it simplifies the calculation and still approximates the distribution for very small values of the mutation rates ϵ. However on the one hand we introduce mutation to obtain an invariant distribution over all states in the *Markov chain* (see 2.4.1). On the other hand we are only looking at the limit of the invariant distribution. In this limiting process the mutation probability goes to zero and therefore we are actually looking at a process in which no mutations occur.

Therefore in this chapter we will focus on the invariant distribution with positive mutation probabilities, because as also Ellison (2000, p. 22) stressed "a characterization of μ^ϵ would be the ideal description of the long run consequences of evolution.".

Instead of computing the exact values we will just simulate the *Markov chain*. By simulating the process a high number of times[1] and counting the periods spent in each state we can approximate[2] for the invariant distribution $\mu(\epsilon)$.

In appendix B.1 some technical informations about the simulation program can be found.

[1]Depending on the mutation rate we use one billion and ten billion rounds for each simulation result.

[2]With this approach we obtain approximations for specific values of ϵ instead of approximations for only very small mutations rates which were obtained by the limit invariant distribution $\mu^\star$.

4.1 Mutation Rate

First we will shortly illustrate the simulation and show the behavior of the probability distribution for decreasing values of ϵ. In figure 4.1 the probability distributions over the state space can be seen for certain mutation rates. In table 4.1 the commonly used parameters are shown. With these parameters we get $\omega^\star = (10, 10)$ as the *Imitation Equilibrium*. The relative time that the process spent in the equilibrium state is denoted by $P(\omega^\star)$. This relative time can also be seen as the probability of being in state $\omega^\star$ if we stop the process at a random period.

Parameters	
Number of rounds	10^9
Price grid	$\gamma = 20$, $\delta = 1$
Initial prices	u.d. over Ω
Mutation prices prob.	u.d. over Γ
Revision prob.	$w = 0.5$
Mutation prob.	specified separately
Transport costs	$t = 10$
Marginal costs	$c = 5$

Table 4.1: Common parameters used for figure 4.1

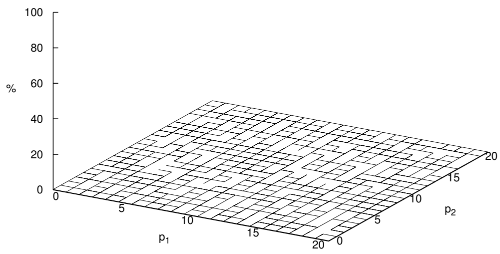
In the figure it can be clearly seen that the probability of being in $\omega^\star$ increases by reducing the mutation rate. Obviously with a mutation rate of one we are getting nearly a uniform distribution over the state space[3]. With a mutation rate of $\epsilon = 10^{-4}$ already almost the whole weight of the distribution is on the equilibrium state.

For the further simulation we will use the mutation rate values $\epsilon = 10^{-2}$ and $\epsilon = 10^{-4}$, because in our opinion these mutation rates represent quite reasonable values for the occurrence of mutation. These mutation rates are already small enough to represent errors or experiments in the game.

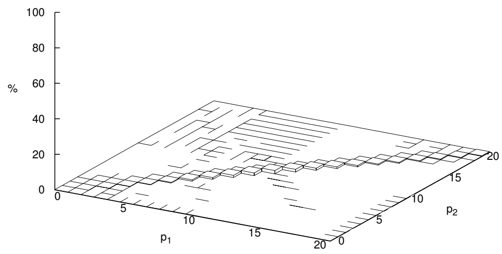
4.2 State Space

In this section we focus on the state space which is in our model the used price grid. We will refine the price grid for fixed mutation rate values and

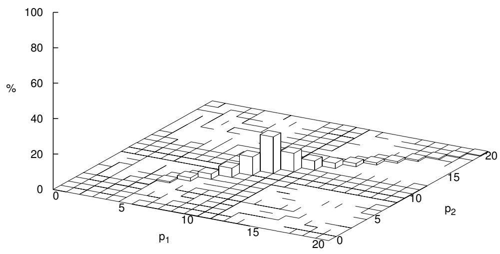
[3] $\gamma = 20$ therefore $|\Omega| = |\Gamma|^2 = (\gamma + 1)^2 = 441$ and so $\forall \omega \in \Omega : P(h_\omega) \approx \frac{1}{441} \approx 0.0023$. We are getting this value for $P(\omega^\star)$ for the mutation rate of one.



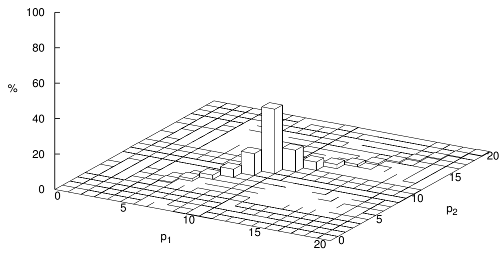
(a) $\epsilon = 1,\ P(\omega^{\star}) = 0.0023$



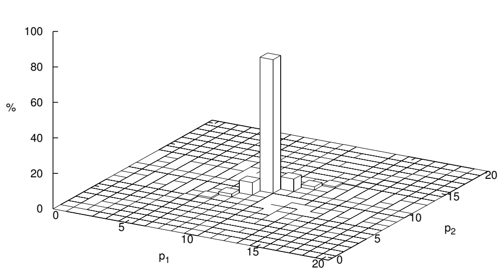
(b) $\epsilon = 0.5,\ P(\omega^{\star}) = 0.0214$



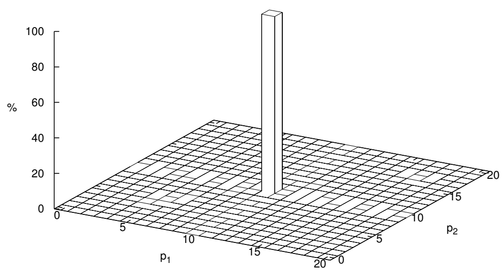
(c) $\epsilon = 0.1,\ P(\omega^{\star}) = 0.2091$



(d) $\epsilon = 0.05,\ P(\omega^{\star}) = 0.3666$



(e) $\epsilon = 10^{-2},\ P(\omega^{\star}) = 0.7539$



(f) $\epsilon = 10^{-4},\ P(\omega^{\star}) = 0.9967$

Figure 4.1: Probability distributions for different mutation rate values

compare the probabilities of being in the *Imitation Equilibrium*.

For the simulation results displayed in table 4.2 we used almost the same parameters as in the previous simulation (stated in table 4.1) except for the used price grid and the mutation rate which vary for each simulation and are listed separately in the result table.[4]

γ	δ	$P(\omega^\star)$ $\epsilon = 10^{-2}$	$P(\omega^\star)$ $\epsilon = 10^{-4}$
10	2.000	0.851958	0.998248
20	1.000	0.753094	0.996804
50	0.400	0.560616	0.992861
100	0.200	0.393368	0.985254
200	0.100	0.247446	0.970592
500	0.040	0.116283	0.931960
1000	0.020	0.062183	0.862690
2000	0.010	0.032084	0.781912
5000	0.004	0.012772	0.599235
10000	0.002	0.006572	0.213496

Table 4.2: $P(\omega^\star)$ in respect to the size of the price grid

Note that the maximum price is 20 units for all grid sizes. Thus we are only refining the price grid. Also the *Imitation Price* of 10 units is in every refinement in the price grid Γ, since γ is an even number in every specified simulation.

Interestingly for the high mutation rate case ($\epsilon = 10^{-2}$) and for γ values around 200 we are already getting probabilities to stay in the equilibrium state smaller than a fourth. This means the process is in the *Long Run Equilibrium* on average only every fourth period. In the case of the low mutation rate we arrive at this point at the a grid size of around 10000.

Thus with a fixed positive mutation rate the actual time spent in a *Long Run Equilibrium* depends considerably on the grid size. Therefore we should not only consider and analyze the implication of the equilibrium state but rather also the implications of the off-equilibrium states because the process spends more time in off-equilibrium states than in the equilibrium state.[6]

[4]To get more robust results we used 10^{10} rounds for the results of the low mutation rate $\epsilon = 10^{-4}$.

[5]You are still reading? You owe me an explanation. I owe you a glass of wine.

[6]However in our model the probable off-equilibrium states have all prices near the *Imitation Price*. In other games there could be a fundamental difference between the equilibrium state and some off-equilibrium states, like in coordination games.

Conclusion

In this paper we have shown that in the *Hotelling Game* imitation behavior leads to more competitive outcomes than the *Nash Equilibrium*.

Interestingly the *Imitation Equilibrium* lies exactly between marginal cost pricing and the *Nash Equilibrium*. In the modification chapter we have seen that this result holds also for modified consumer distributions. As well we have seen that by introducing asymmetric cost structures for the player the uniqueness of the *Imitation Equilibrium* gets lost.

In the last part we simulated the *Markov Process* and checked the robustness of the *Imitation Equilibrium* with positive mutation rates, since the definition of the *Imitation Equilibrium* is based on a limiting process. There we have seen that the grid size influences considerable the long run behavior of the process.

Appendix A

Mathematical Proofs

A.1 Proof of proposition 3

To prove the proposition 3 we have to show that the best response to $p^\star = c + \frac{t}{\alpha}$ is $p^\star$. For simplicity we will set the marginal cost c in this proof to zero. The proof would be analogous with marginal costs different from zero. Mathematically expressed we have to show that $\operatorname{argmax}_p \pi_i^\alpha(p, \frac{t}{\alpha}) = \frac{t}{\alpha}$. Lets start:

$$\begin{aligned}
&\operatorname*{argmax}_p \pi_i^\alpha(p, \frac{t}{\alpha}) = \\
&\operatorname*{argmax}_p p\, s_i^\alpha(p, \frac{t}{\alpha}) = \\
&\operatorname*{argmax}_p p\, F_\alpha\big(s_i(p, \frac{t}{\alpha})\big)
\end{aligned}$$

Because of the piecewise definition of $F_\alpha(x)$ we have to consider four different cases:

Case 1: $p \leq \frac{t}{\alpha} - t$

If the price of firm i is smaller than $\frac{t}{\alpha} - t$ the firm gets the whole market share $s_i(p, \frac{t}{\alpha}) = 1$. So the maximization problem on this interval is simply $\operatorname{argmax}_p p$. The maximizer is therefore $p^{1\star} = \frac{t}{\alpha} - t$. In this case the maximum profit and the maximizer coincide.[1]

[1] This case is irrelevant in the proof with zero marginal costs and $\alpha > 1$, because we are not allowing negative prices. However this case would for example be necessary with

Case 2: $p \geq \frac{t}{\alpha} + t$

This is the contrary case to the first case. The price is too high and no consumer will buy from firm i. The market share is zero and so are the profits. Maximizers are in this case all $p \geq \frac{t}{\alpha} + t$ therefore the maximizers $p^{2\star}$ on this interval are the whole interval and the maximum profit reached is zero.

Case 3: $\frac{t}{\alpha} \leq p < \frac{t}{\alpha} + t$

The price of firm i is higher than the price of the other firm but still small enough to have some market share. The unweighted market share lies in the interval $(0, \frac{1}{2}]$ so in this case we can use for $F_\alpha(x) = \frac{1}{2}(2x)^\alpha$ and for $f_\alpha(x) = \alpha(2x)^{\alpha-1}$.

We will maximize this function on that interval, so we need the first derivative of the profit function in respect to p:

$$\frac{d\,\pi_i^\alpha(p, \frac{t}{\alpha})}{dp} = \frac{d\Big(p\,F_\alpha\big(s_i(p, \frac{t}{\alpha})\big)\Big)}{dp} \tag{A.1}$$

$$= F_\alpha\big(s_i(p, \frac{t}{\alpha})\big) + p f_\alpha\big(s_i(p, \frac{t}{\alpha})\big)\frac{d}{dp}\Big(s_i(p, \frac{t}{\alpha})\Big) \tag{A.2}$$

$$= F_\alpha\big(s_i(p, \frac{t}{\alpha})\big) - \frac{p}{2t} f_\alpha\big(s_i(p, \frac{t}{\alpha})\big) \tag{A.3}$$

By setting the derivative to zero we get the turning point:

$$\begin{aligned}
\frac{d\,\pi_i^\alpha(p, \frac{t}{\alpha})}{dp} &= 0 \\
F_\alpha\big(s_i(p, \frac{t}{\alpha})\big) &= \frac{p}{2t} f_\alpha\big(s_i(p, \frac{t}{\alpha})\big) \\
\frac{1}{2}\big(2s_i(p, \frac{t}{\alpha})\big)^\alpha &= \frac{p\alpha}{2t}\big(2s_i(p, \frac{t}{\alpha})\big)^{\alpha-1} \\
s_i(p, \frac{t}{\alpha}) &= \frac{p\alpha}{2t} \\
\frac{1}{2t}\big(\frac{t}{\alpha} + t - p\big) &= \frac{p\alpha}{2t} \\
p(1+\alpha) &= t\big(1 + \frac{1}{\alpha}\big) \\
p &= \frac{t}{\alpha}
\end{aligned}$$

$c > \frac{t}{\alpha}(\alpha - 1)$.

To check whether this is a maximum or a minimum we could calculate the second derivative. We will not do this here, because the second derivative[2] would contain the first derivative of the density function $f_\alpha(x)$ which does not exist at the point $x = \frac{t}{\alpha}$. Instead we will show that the first derivative is negative on the whole interval except for the crucial point $\frac{t}{\alpha}$.

$$
\begin{aligned}
\frac{d\,\pi_i^\alpha(p, \frac{t}{\alpha})}{dp} &= F_\alpha\big(s_i(p, \frac{t}{\alpha})\big) - \frac{p}{2t} f_\alpha\big(s_i(p, \frac{t}{\alpha})\big) \\
&= \frac{1}{2}\big(2s_i(p, \frac{t}{\alpha})\big)^\alpha - \frac{p\alpha}{2t}\big(2s_i(p, \frac{t}{\alpha})\big)^{\alpha-1} \\
&= \big(2s_i(p, \frac{t}{\alpha})\big)^{\alpha-1}\Big(s_i(p, \frac{t}{\alpha}) - \frac{p\alpha}{2t}\Big) \\
&= \big(2s_i(p, \frac{t}{\alpha})\big)^{\alpha-1}\Big(\frac{\frac{t}{\alpha} + t - p}{2t} - \frac{p\alpha}{2t}\Big) \\
&= \frac{\big(2s_i(p, \frac{t}{\alpha})\big)^{\alpha-1}}{2t}\Big(\frac{t}{\alpha} + t - p - p\alpha\Big) \\
&= \underbrace{\frac{(1+\alpha)\big(2s_i(p, \frac{t}{\alpha})\big)^{\alpha-1}}{2t}}_{>0}\Big(\frac{t}{\alpha} - p\Big)
\end{aligned}
$$

The expression $\big(\frac{t}{\alpha} - p\big)$ is negative in the considered interval except for $p = \frac{t}{\alpha}$. So we have in this interval the maximizer $p^{3\star} = \frac{t}{\alpha}$ and a maximum profit of $\frac{t}{2\alpha}$.

Case 4: $\frac{t}{\alpha} - t < p \leq \frac{t}{\alpha}$

This case represents a price of firm i which is lower than the price of firm j but still high enough for the other firm to have a positive market share. Here we can use the $F_\alpha(x) = 1 - \frac{1}{2}\big(2(1-x)\big)^\alpha$ and $f_\alpha(x) = \alpha\big(2(1-x)\big)^{\alpha-1}$ definition of the distribution function.

---

[2] $\frac{d^2\,\pi_i^\alpha(p, \frac{t}{\alpha})}{dp^2} = -\frac{1}{2t}\bigg(2f_\alpha\big(s_i(p, \frac{t}{\alpha})\big) + p\,\frac{d}{dp}\Big(f_\alpha\big(s_i(p, \frac{t}{\alpha})\big)\Big)\bigg)$

Setting $p = \frac{t}{\alpha}$ in the first derivative leads to:

$$
\begin{aligned}
&F_\alpha\big(s_i(\frac{t}{\alpha},\frac{t}{\alpha})\big) - \frac{\frac{t}{\alpha}}{2t} f_\alpha\big(s_i(\frac{t}{\alpha},\frac{t}{\alpha})\big) \\
&= 1 - \frac{1}{2}\big(2(1 - s_i(\frac{t}{\alpha},\frac{t}{\alpha}))\big)^\alpha - \frac{1}{2\alpha}\alpha\big(2(1 - s_i(\frac{t}{\alpha},\frac{t}{\alpha}))\big)^{\alpha-1} \\
&= 1 - \frac{1}{2}\big(2(1-\frac{1}{2})\big)^\alpha - \frac{1}{2}\big(2(1-\frac{1}{2})\big)^{\alpha-1} \\
&= 1 - \frac{1}{2}(1)^\alpha - \frac{1}{2}(1)^{\alpha-1} = 0
\end{aligned}
$$

Same procedure as before: If we can show that the first derivative is positive on the whole interval except for the point $p = \frac{t}{\alpha}$ we have found the maximum on this interval.

$$
\begin{aligned}
\frac{d\,\pi_i^\alpha(p,\frac{t}{\alpha})}{dp} &= F_\alpha\big(s_i(p,\frac{t}{\alpha})\big) - \frac{p}{2t} f_\alpha\big(s_i(p,\frac{t}{\alpha})\big) \\
&= 1 - \frac{1}{2}\Big(2\big(1 - s_i(p,\frac{t}{\alpha})\big)\Big)^\alpha - \frac{p\alpha}{2t}\Big(2\big(1 - s_i(p,\frac{t}{\alpha})\big)\Big)^{\alpha-1} \\
&= 1 - \frac{1}{2}\bigg(\Big(2s_i(\frac{t}{\alpha},p)\Big)^\alpha + \frac{p\alpha}{t}\Big(2s_i(\frac{t}{\alpha},p)\Big)^{\alpha-1}\bigg) \\
&= 1 - \frac{1}{2}\Big(2s_i(\frac{t}{\alpha},p)\Big)^{\alpha-1}\Big(2s_i(\frac{t}{\alpha},p) + \frac{p\alpha}{t}\Big)
\end{aligned}
$$

Let I be the open interval $(\frac{t}{\alpha} - t, \frac{t}{\alpha})$. It is trivial to see that the expression $2s_i(\frac{t}{\alpha},p)$ is smaller than one on the whole interval I[3]. If we additionally assume that $\alpha \geq 1$ we also can conclude that $\big(2s_i(\frac{t}{\alpha},p)\big)^{\alpha-1} \leq 1$. Thus we now need to show only that $\big(2s_i(\frac{t}{\alpha},p) + \frac{p\alpha}{t}\big) < 2$ on the interval I. We have already considered the first term which is always smaller than one. The same holds for the the second term $\frac{p\alpha}{t}$ because this expression is increasing in respect to p and only for the upper bound of the interval I: $p = \frac{t}{\alpha}$ the term reaches one. Therefore we also get the same maximizer as in case 3, so $p^{4\star} = p^{3\star}$.

Comparing all maximum values of the maximizers in each interval leads to the global maximizer $p^\star = \frac{t}{\alpha}$ for $\forall \alpha \in [1,\infty)$. Hence we obtain as best response to the price $p_j = \frac{t}{\alpha}$ the price $p_i = \frac{t}{\alpha}$. Due to the symmetric property of this game this is a fixed point and therefore a *Nash Equilibrium*. □

[3]The price of the other firm is lower.

Appendix B

Simulation

B.1 Implementation

The simulation is implemented in an object oriented manner in the programming language `C++`. It is a program designed for the console which uses command line arguments to set the values of the parameters. The program outputs data in such a format that this output can be used by `gnuplot`[1] to create the probability distribution figures that are used in the thesis.

B.2 Additional Evaluations

In this part of the appendix some simulation outputs are shown which are referenced in the text and give a visual impression (the probability distribution over the states) of some of the presented analytic results.

Common parameters The initial price is uniformly distributed over the state space Ω. Also if the revision opportunity arises the firms will mutate to any price in the price grid Γ with the same probability. If not differently stated in the parameters we assume uniformly distributed consumers, so in terms of the distribution modification the parameter α is set to one.

[1]see http://www.gnuplot.info

B.2.1 Asymmetric players

Parameters	
Rounds	one billion
Price grid	$\gamma = 40$, $\delta = 1$
Mutation prob.	$\epsilon = 0.01$
Revision prob.	$w = 1$
Transport costs	$t = 10$
Marginal costs	$c_i = 14$, $c_j = 16$

Table B.1: Simulation, Parameters ($c_i \neq c_j$)

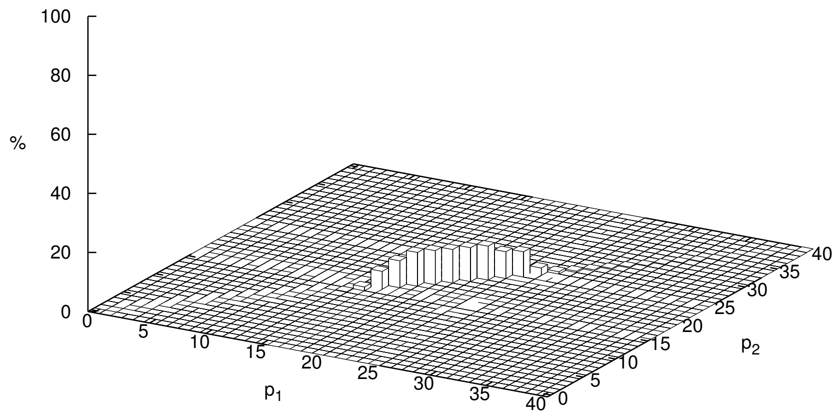


Figure B.1: Simulation, Prob. distribution ($c_i \neq c_j$)

B.2.2 No transport costs

Marginal costs in the price grid

Parameters	
Rounds	one billion
Price grid	$\gamma = 20$, $\delta = 1$
Mutation prob.	$\epsilon = 0.01$
Revision prob.	$w = 1$
Transport costs	$t = 0$
Marginal costs	$c = 5$

Table B.2: Simulation, Parameters ($t = 0$, $c \in \Gamma$)

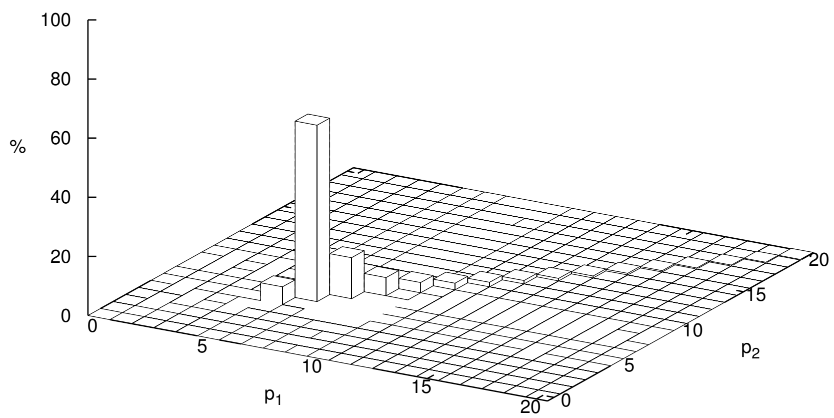


Figure B.2: Simulation, Prob. distribution ($t = 0$, $c \in \Gamma$)

Marginal costs not in the price grid

Parameters	
Rounds	one billion
Price grid	$\gamma = 10$, $\delta = 2$
Mutation prob.	$\epsilon = 0.01$
Revision prob.	$w = 1$
Transport costs	$t = 0$
Marginal costs	$c = 5$

Table B.3: Simulation, Parameters ($t = 0$, $c \notin \Gamma$)

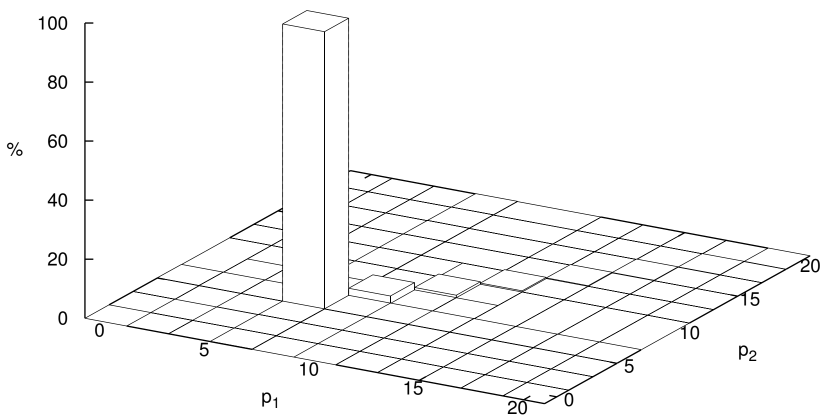


Figure B.3: Simulation, Prob. distribution ($t = 0$, $c \notin \Gamma$)

B.2.3 Non-uniform Consumer Distribution

Centered distribution ($\alpha = 4$)

Parameters	
Rounds	one billion
Price grid	$\gamma = 20$, $\delta = 2$
Mutation prob.	$\epsilon = 0.01$
Revision prob.	$w = 1$
Transport costs	$t = 16$
Marginal costs	$c = 8$
Consumer dist.	$\alpha = 4$

Table B.4: Simulation, Parameters ($\alpha = 4$)

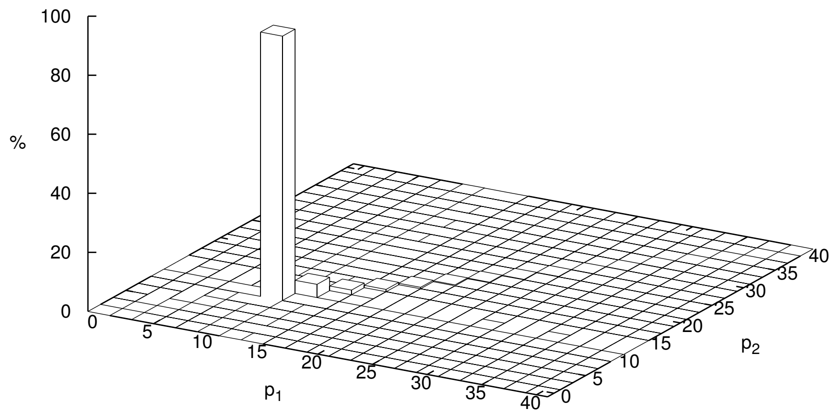


Figure B.4: Simulation, Prob. distribution ($\alpha = 4$)

Non-centered distribution ($\alpha = 0.25$)

Parameters	
Rounds	one billion
Price grid	$\gamma = 20$, $\delta = 2$
Mutation prob.	$\epsilon = 0.01$
Revision prob.	$w = 1$
Transport costs	$t = 16$
Marginal costs	$c = 8$
Consumer dist.	$\alpha = 0.25$

Table B.5: Simulation, Parameters ($\alpha = 0.25$)

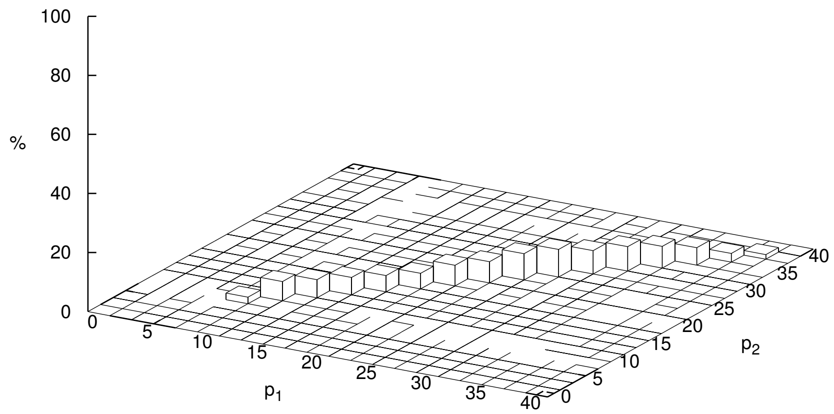


Figure B.5: Simulation, Prob. distribution ($\alpha = 0.25$)

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Abstract

English

Vega-Redondo has shown in the paper “The Evolution of Walrasian behavior” that imitation leads to *Walrasian Prices* in the symmetric *Cournot Game*. Instead of the *Cournot Game* we choose as underlying game the *Hotelling Game*. Like Vega-Redondo we use the ‘imitation of the best’ dynamics and the *Freidlin and Wentzell* approach to determine the *Long Run Equilibria* (*LRE*). As *LRE* under imitation we obtain interestingly the average of the *Nash Equilibrium* and marginal cost pricing. Later on we test the robustness of the obtained *LRE*. On the one hand we check this by slightly altering the assumption of the underlying *Hotelling Game* and consider their implication on the *LRE* under imitation. On the other hand we simulate the *Markov Process* with positive mutation rates and focus on the probability distribution over the state space.

German

Vega-Redondo zeigte in dem Artikel "The Evolution of Walrasian behavior", dass Imitation in einem symmetrischen *Cournot-Oligopol* zu den *Walras Preisen* führt. Anstelle des *Cournot-Oligopol* verwenden wir das *Hotelling-Modell* als das zugrunde liegende Modell. Ebenso wie Vega-Redondo verwenden wir die 'Imitation des Besten' als Dynamik und nutzen die *Freidlin und Wentzell* Methode zur Bestimmung der *Langzeitgleichgewichte* (*LZG*). Interessanterweise erhalten wir hier als *LZG* den Mittelwert des *Nash-Gleichgewichtspreises* und des Grenzkostenpreises. Im Anschluss darauf testen wir die Robustheit des erhaltenen *LZG*es. Dies überprüfen wir einerseits durch kleine Modifikationen der Annahmen des zugrunde liegenden Modells und den daraus resultierenden Auswirkungen auf das Imitations-*LZG*. Andererseits simulieren wir den *Markov Prozess* mit positiven Mutationsraten und betrachten die Wahrscheinlichkeitsverteilungen über dem Zustandsraum.

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